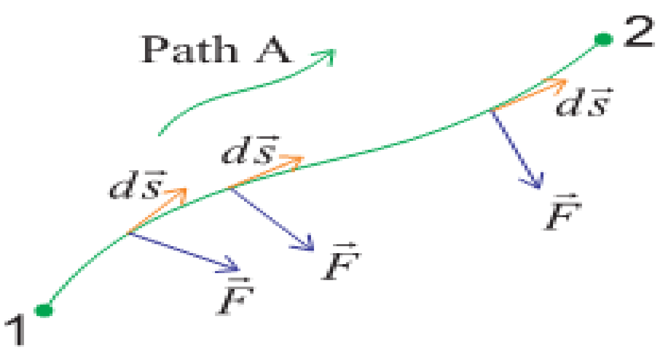
Electricity & magnetism-1

Electric potential energy and electric potential

Electric potential energy

Electric force is a **conservative force**



Work done by the electric force $\vec{F}$ as a
charge moves an infinitesimal distance d.
along $Path\ A = dW$

Note: $d\vec{s}$ is in the *tangent* direction of the curve of $Path\ A$.

$$dW = \vec{F} \cdot d\vec{s}$$

$\therefore$ Total work done W by force $\vec{F}$ in moving the particle from Point 1 to Point 2

$$\boxed{W = \underset{\text{Path A}}{\int_1^2} \vec{F} \cdot d\vec{s}}$$

$$\underset{\text{Path A}}{\int_1^2} \quad = \quad Path\ Integral$$

$$= \quad \text{Integration over Path A from Point 1 to Point 2.}$$

Continue..

For an infinitesimal displacement $d\mathbf{s}$ of a charge, the work done by the electric field on the charge is $\mathbf{F} \cdot d\mathbf{s} = q_0\mathbf{E} \cdot d\mathbf{s}$. As this amount of work is done by the field, the potential energy of the charge–field system is changed by an amount $dU = -q_0\mathbf{E} \cdot d\mathbf{s}$. For a finite displacement of the charge from point A to point B, the change in potential energy of the system $\Delta U = U_B - U_A$ is

$$\Delta U = -q_0 \int_A^B \mathbf{E} \cdot d\mathbf{s}$$

The integration is performed along the path that q_0 follows as it moves from A to B. Because the force $q_0\mathbf{E}$ is conservative, **this line integral does not depend on the path taken from A to B.**

Electric potential

Consider a charge q at center, we consider its effect on test charge q_0

DEFINITION: We define electric potential V so that

$$\boxed{\Delta V = \frac{\Delta U}{q_0} = \frac{-\Delta W}{q_0}}$$

($\therefore$ V is the P.E. per unit charge)

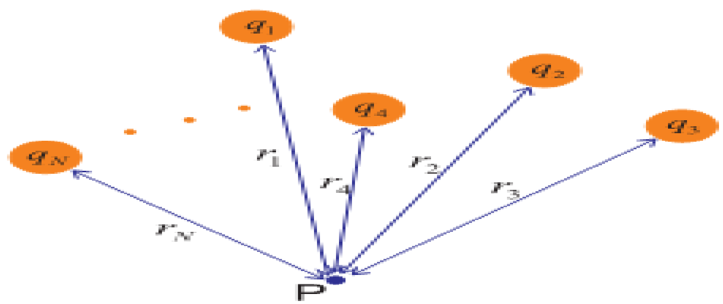
- Similarly, we take $V(r = \infty) = 0$.
- Electric Potential is a **scalar**.
- Unit: $Volt(V) = Joules/Coulomb$
- For a single point charge:

$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r}}$$

- Energy Unit: $\Delta U = q\Delta V$

$$electron - Volt(eV) = \underbrace{1.6 \times 10^{-19}}_{\text{charge of electron}} \; J$$

<u>Potential For A System of Charges</u>



For a total of N point charges, the potential V at any point P can be derived from the **principle of superposition**.

Recall that potential due to q_1 at point P: $V_1 = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1}{r_1}$

$\therefore$ Total potential at point P due to N charges:

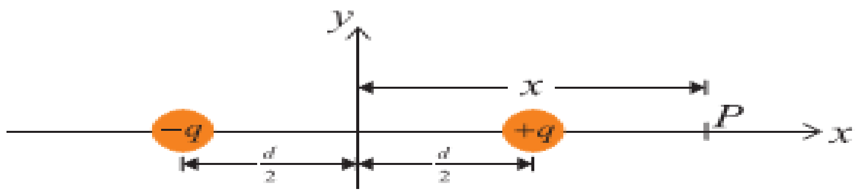
$$\begin{aligned} V &= V_1 + V_2 + \cdots + V_N \quad \text{(principle of superposition)} \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2} + \cdots + \frac{q_N}{r_N}\right] \end{aligned}$$

Continue..

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i}}$$

Note: For $\vec{E}, \vec{F}$, we have a sum of vectors
For V, U, we have a sum of scalars

Example: Potential of an electric dipole



$\vec{p} = q\vec{d}$

Consider the potential point P at distance $x >$ from dipole.

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{+q}{x-\frac{d}{2}} + \frac{-q}{x+\frac{d}{2}}\right.$$

Special Limiting Case: $x \gg d$

$$\frac{1}{x \mp \frac{d}{2}} = \frac{1}{x}\cdot\frac{1}{1 \mp \frac{d}{2x}} \simeq \frac{1}{x}\left[1 \pm \frac{d}{2x}\right]$$

$$\therefore \quad V = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{x}\left[1 + \frac{d}{2x} - (1 - \frac{d}{2x})\right]$$

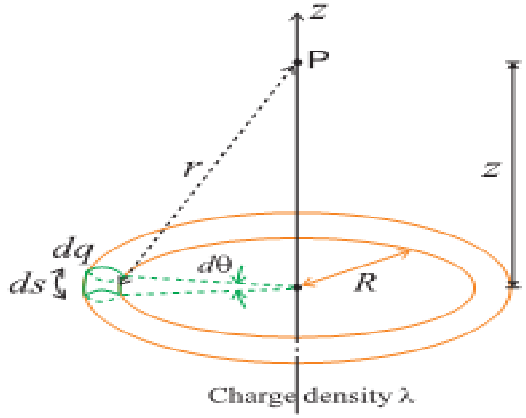
$$V = \frac{p}{4\pi\epsilon_0 x^2} \qquad \text{(Recall } p = qd\text{)}$$

For a point charge $\quad E \propto \frac{1}{r^2} \quad V \propto \frac{1}{r}$

For a dipole $\quad E \propto \frac{1}{r^3} \quad V \propto \frac{1}{r^2}$

For a quadrupole $\quad E \propto \frac{1}{r^4} \quad V \propto \frac{1}{r^3}$

Electric potential for uniform charge rod



Length of the infinitesimal ring elemen $= ds = Rd\theta$

$$\therefore \quad \text{charge } dq = \lambda\, ds = \lambda R\, d\theta$$

$$dV = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\theta}{\sqrt{R^2 + z^2}}$$

The integration is around the entire ring.

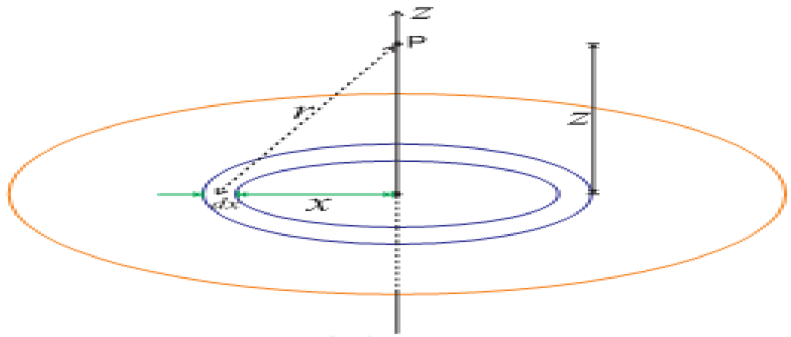
$$\begin{aligned} \therefore \quad V &= \int_{\text{ring}} dV \\ &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\theta}{\sqrt{R^2 + z^2}} \\ &= \frac{\lambda R}{4\pi\epsilon_0\sqrt{R^2 + z^2}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \end{aligned}$$

Total charge on the ring $= \lambda \cdot (2\pi R)$

$$V = \frac{Q}{4\pi\epsilon_0\sqrt{R^2 + z^2}}$$

LIMITING CASE: $\quad z \gg R \;\Rightarrow\; V = \dfrac{Q}{4\pi\epsilon_0\sqrt{z^2}} = \dfrac{Q}{4\pi\epsilon_0|z|}$

Electric potential for disk of charge



Total charge = Q
Charge density = σ

Using the **principle of superposition**, we will find the potential of a disk of uniform charge density by integrating the potential of *concentric rings.*

$$\therefore \quad dV = \frac{1}{4\pi\epsilon_0}\int_{\text{disk}} \frac{dq}{r}$$

Ring of radius x: $\quad dq = \sigma\, dA = \sigma\,(2\pi x dx)$

$$\begin{aligned}
\therefore \quad V &= \int_0^R \frac{1}{4\pi\epsilon_0}\cdot\frac{\sigma 2\pi x\, dx}{\sqrt{x^2+z^2}} \\
&= \frac{\sigma}{4\epsilon_0}\int_0^R \frac{d(x^2+z^2)}{(x^2+z^2)^{1/2}} \\
V &= \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2}-\sqrt{z^2}\right) \\
&= \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2}-|z|\right)
\end{aligned}$$

Recall:
$$|x| = \begin{cases} +x; & x \geq 0 \\ -x; & x < 0 \end{cases}$$

Limiting Case:

(1) If $|z| \gg R$

$$\begin{aligned}
\sqrt{z^2+R^2} &= \sqrt{z^2\left(1+\frac{R^2}{z^2}\right)} \\
&= |z|\cdot\left(1+\frac{R^2}{z^2}\right)^{\frac{1}{2}} && (\ (1+x)^n \approx 1+nx \text{ if } x \ll 1\ ) \\
&\simeq |z|\cdot\left(1+\frac{R^2}{2z^2}\right) && \left(\ \frac{|z|}{z^2}=\frac{1}{|z|}\ \right)
\end{aligned}$$

$\therefore$ At large z, $V \simeq \frac{\sigma}{2\epsilon_0}\cdot\frac{R^2}{2|z|} = \frac{Q}{4\pi\epsilon_0|z|}$ (like a point charge)
where Q = total charge on disk $= \sigma\cdot\pi R^2$

Relation between E and V

(A) To get V from $\vec{E}$:
Recall our definition of the potential V:

$$\Delta V = \frac{\Delta U}{q_0} = -\frac{W_{12}}{q_0}$$

where ΔU is the change in P.E.; W_{12} is the work done in bringing charge q_0 from point 1 to 2.

$$\therefore \quad \Delta V = V_2 - V_1 = \frac{-\int_1^2 \vec{F} \cdot d\vec{s}}{q_0}$$

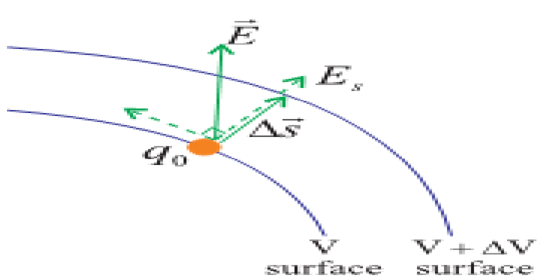
However, the definition of E-field: $\quad \vec{F} = q_0 \vec{E}$

$$\therefore \quad \boxed{\Delta V = V_2 - V_1 = -\int_1^2 \vec{E} \cdot d\vec{s}}$$

Note: The integral on the right hand side of the above can be calculated *along any path from point 1 to 2.* **(Path-Independent)**

Convention: $V_\infty = 0 \quad \Rightarrow \quad \boxed{V_P = -\int_\infty^P \vec{E} \cdot d\vec{s}}$

(B) To get $\vec{E}$ from V:



(i.e. Potential = V on the surface)

Again, use the definition of V:

$$\Delta U = q_0 \Delta V = \underbrace{-W}_{\text{Work done}}$$

However,

$$W = \underbrace{q_0 \vec{E}}_{\text{Electric force}} \cdot \Delta\vec{s} = q_0 E_s \Delta s$$

where E_s is the E-field component along the path $\Delta\vec{s}$.

$$\therefore \quad q_0 \Delta V = -q_0 E_s \Delta s$$

Continue…

$$\therefore \quad E_s = -\frac{\Delta V}{\Delta s}$$

For infinitesimal Δs,

$$\therefore \quad \boxed{E_s = -\frac{dV}{ds}}$$

Note: (1) Therefore the E-field component along *any direction* is the negtive derivative of the potential *along the same direction*.

(2) If $d\vec{s} \perp \vec{E}$, then $\Delta V = 0$

(3) ΔV is biggest/smallest if $d\vec{s} \parallel \vec{E}$

Generally, for a potential $V(x, y, z)$, the relation between $\vec{E}(x, y, z)$ and V is

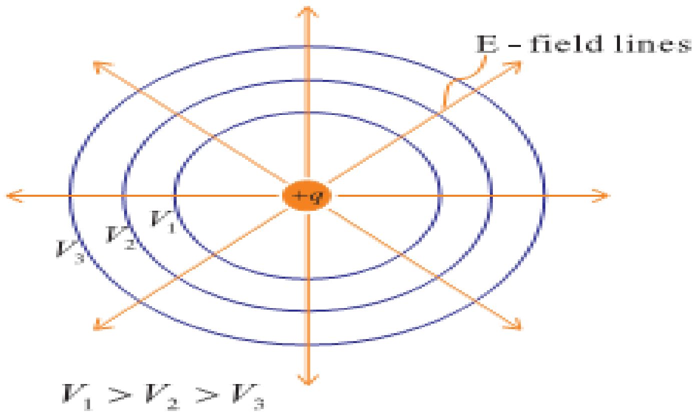
$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are **partial derivatives**

For $\frac{\partial}{\partial x}V(x, y, z)$, everything y, z are treated like a *constant* and we only take derivative with respect to x.

Equipotential surface

Equipotential surface is a surface on which the *potential is constant*.
$\Rightarrow \quad (\Delta V = 0)$



$$V(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{+q}{r} = const$$

$\Rightarrow \quad r = const$

$\Rightarrow$ Equipotential surfaces are *circles/spherical surfaces*

Note: (1) A charge can move freely on an equipotential surface without any work done.

(2) The **electric field lines** must be *perpendicular* to the **equipotential surfaces.** (Why?)
On an equipotential surface, $V = constant$
$\Rightarrow \quad \Delta V = 0 \quad \Rightarrow \quad \vec{E} \cdot d\vec{l} = 0$, where $d\vec{l}$ is *tangent* to equipotential surface
$\therefore \quad \vec{E}$ must be *perpendicular* to equipotential surfaces.

All sample problems of ch# 28